Poincaré limit cycles and the theory of self-sustaining oscillations
(Les cycles limites de Poincaré et la théorie des oscillations autoentretenues)

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1. In recent years so-called *self-sustaining* oscillations have aroused an increasingly keen interest in several areas of the natural sciences. These oscillations are governed by differential equations which differ from those studied in mathematical physics and classical mechanics. The systems in which these phenomena arise are non-conservative, and sustain their oscillations by drawing energy from non-periodic sources.

We can cite, for the case of partial differential equations, the ancient problem of a vibrating string excited by a bow, as well as the problem of the Cepheids, addressed by Eddington[1]; for the case of ordinary differential equations we have, in mechanics the Froude pendulum[2], in physics the triode oscillator[3]; in chemistry periodic reactions[4]; and similar problems arise in biology[5].

2. Consider the simplest case of the self-oscillations that arise, in mechanics and physics, in a system with one degree of freedom; in chemistry in a reaction between two substances; in biology when two animal species co-exist. These systems can be represented by two simultaneous differential equations:

$$\text{(A)} \qquad \frac{dx}{dt} = P(x, y); \qquad \frac{dy}{dt} = Q(x, y).$$

It is well-known that the steady-state solutions of such a system can be of two types: either constant or periodic in t. We require, on the basis of the study of actually observed phenomena of this type, that the periodic motions under consideration are stable with respect to sufficiently small arbitrary variations in (1) initial conditions[6] and (2) the second elements of equations A[7].

It may easily be shown that, to periodic motions satisfying these conditions, there correspond, in the xy plane, isolated closed curves, approached in spiral fashion by neighbouring solutions from the interior or the exterior (for increasing t). As a result, *self-oscillations arising in systems characterised by equations of type A correspond mathematically to stable Poincaré limit cycles*[8].

It is thus clear that the period and amplitude of the steady-state oscillations are independent of the intial conditions. Discussion of differential equations relating to real examples show that they exhibit limit cycles, which define the steady state motion.

3. The general theory[9] of the integral curves [solutions] of equations of type A allows, in many cases, the qualitative study of these equations and to draw conclusions as to the existence, number,

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[1] Eddington, *The internal constitution of stars*, p. 200 (Cambridge, 1926)
[2] Lord Rayleigh, *The theory of sound*, London, 1, 1894, p. 212
[3] See, for example, Van der Pol, *Phil. Mag.*, 7th series, 2, 1926, p. 978
[4] See, for example, Kremann, *Die periodischen Erscheinungen in der Chemie*, p. 124 (Stuttgart, 1913)
[5] Lotka, *Elements of physical biology*, p. 88 (Baltimore, 1925). See also the recent researches of Monsieur Volterra.
[6] See Liapunow, *Problème général de la stabilité du mouvement* (*Ann. de la Faculté des Sciences de Toulouse*, 9, 1907, p. 209)
[7] See Bieberback, *Differentialgleichungen*, p. 68 (Berlin, 1926)
[8] Poincaré, *Oeuvres*, 1, p. 53 (Paris, 1928)
[9] Pincaré, *loc. cit.* Bendixson, *Acta mathematica*, 24, 1900, p.1

and stability of the limit cycles. The quantitative solution of the problem, which consists in expressing x and y as function of time, can only be obtained easily for the case of small parameter values[10]. Consider, as an example, the particular case[11] where equations A are

(B) $$\frac{dx}{dt} = y + \mu\, f(x, y; \mu); \qquad \frac{dy}{dt} = -x + \mu\, g(x, y; \mu).$$

where μ is a real parameter which we can choose to be sufficiently small. When $\mu = 0$, equations B have a solution $x = R \cos t$, $y = -R \sin t$; the solutions form, in the xy plane, a family of circles. Following Poincaré's methods, it can be seen that for sufficiently small $\mu \neq 0$, the xy plane contains only isolated closed curves, near to circles with radii defined by the equation

(C) $$\int_0^{2\pi} [f(R\cos\xi; -R\sin\xi; 0)\cos\xi - g(R\cos\xi - R\sin\xi\,; 0)\sin\xi]\,d\xi = 0$$

These closed curves correspond to stable, steady-state motion where the following condition is fulfilled:

(D) $$\int_0^{2\pi} [f_x(R\cos\xi; -R\sin\xi; 0) + g'_y(R\cos\xi - R\sin\xi\,; 0)]\,d\xi < 0$$

The correction to be made to the fundamental period 2π, and the expressions for x and y, are obtained in the form of ordered series in powers of μ, convergent for sufficiently small values of μ.

4. The theory of self-oscillations, until now almost exclusively supported by non rigorous methods, is thus put on a solid mathematical footing, at least in the simplest case.

Electrical self-oscillations are the most accessible to experimental study. It is certain that a range of characteristic phenomena accompanying these oscillations[12] will also be found in mechanical or chemical self-oscillatory systems.

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[10] Poincaré, Les méthodes nouvelles de la mécanique céleste, 1, p.89 (Paris, 1892)

[11] This case is of great physical interest: sinusoidal self-oscillations in systems of one degree of freedom (triode oscillator, for example) can be reduced to it.

[12] For example, the phenomenon that the Germans call *Mitnehmen* (see H. Barkhausen, Elektronen-Röhren, 3, p. 32, Leipzig, 1929)