On the Dynamics of Automatic Gain Controllers

(Communication from the Central Laboratory of the Siemens & Haske AG Wernerwerk.)

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1. Automatic Gain Controllers

In recent years, devices for the automatic control of gain have increased in importance in various areas of amplifier technology. One class of such devices is based on the following principle. A portion of the output signal current of a valve amplifier is extracted, amplified and fed to a rectifier. The resulting rectified signal voltage is then used to vary the grid voltage of an amplifier valve. In this manner an increase in output power leads to a reduction in gain.

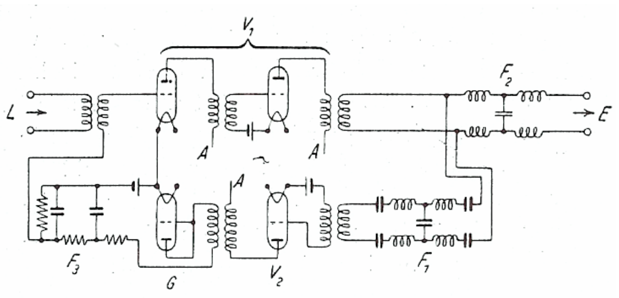

Figure 1. Example of a gain controller

Figure 1 is an example of such a control circuit. The input current at L, deriving from a given transmission system, will consist of signal components together with an a.c. control component whose frequency lies outside the frequency band of the signal components. For speech transmission, for example, the signal frequencies might extend from 300 to 2500 Hz, while the control oscillation, whose amplitude at the transmitter will be held constant, might have a frequency of, say, 3000 Hz. Assuming that variations in the transmission medium affect the control oscillation in the same way as the signal components, then the gain of the entire transmission section up to the output E of the amplifier in Figure 1 will remain constant if the amplifier gain is constantly adjusted in such a way that the control oscillation at the output of the amplifier has a constant amplitude. To this end, the control oscillation is filtered out by means of the wave filter F_1, that passes only frequencies in the region of 3000 Hz, and fed to the amplifier V_2. The voltage output of the rectifier G (negative with respect to the cathode) is cleaned of harmonic components by the filter F_3 and applied to the grid of the first tube of V_1. By appropriate sizing of individual circuit elements the change in grid voltage can lead to a reduction in gain which virtually neutralizes any increase in the control oscillation at the output of amplifier V_1. The filter F_2 prevents the control oscillation from reaching the receiver itself. Such devices are employed to compensate for fading in shortwave transmissions, and will be reported later.

Related devices are amplitude limiters[1] and arrangements for automatic volume control in radio receivers, as have been described by Bruce Wheeler[2] and others, for example. In such cases modification of the gain is sometimes achieved by means other than varying grid voltage: for example, by the additional magnetisation of a choke or a transformer, or by a voltage divider in the amplifier actuated electromagnetically or using a motor.

It is a well-known characteristic of such control arrangements that the variables to be held constant become oscillatory if too tight a control is attempted. Stable equilibrium is possible only under certain conditions. This paper will investigate these conditions in detail and, on the basis of a general representation of the dynamics of this type of control arrangement, will present simple approximation formulae for the estimation of stability conditions.

2. The principle of the continuous, indirect controller

The principle common to the control arrangements described above can be represented by Figure 2.

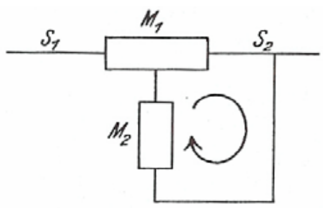

Figure 2. Diagram of the controller

The control arrangement consists in general of a "transmission system" M_1 and a "control system" M_2. The transmission system M_1 represents the relationship between a system variable S_1, that can vary generally with time, and a dependent system variable S_2, that is to be held constant by means of the control arrangement. If the control system M_2 were not present, then in the steady state a relationship would exist between variables S_1 and S_2 that can be written in the form

$$S_2 = AS_1 \qquad (1)$$

where A is the "transmission factor" of the system M_1. A itself can in general depend on S_1.

The action of the control system M_2 consists in modifying the transmission factor A in a way dependent on the variable S_2 such that an increase in S_2 results in a reduction in A and vice versa. Now, the class of controllers to be considered is characterised by a transmission factor A that is a continuous function of S_2,

[1] H. F. Mayer, *ENT*, Vol. 5, 1928, p.468
[2] *Proc. Inst. Radio Eng.*, Vol. 16, 1928, p.30

$$A = f(S_2) \tag{2}$$

In order to achieve control, $f(S_2)$ must be a descending function in the region of interest – that is, with a negative derivative.

In the case of the gain controller, the variable S_2 could be the output voltage, and S_1 an emf or the field strength in the neighbourhood of the receiving antenna. The transmission factor A is determined by the amplifier gain; it is modified by the variation in the grid voltage of the amplifier tube as a function of the output voltage S_2 according to Eq 2.

Figure 2 can be applied to a range of other control arrangements, as can easily be seen. In the simplest case of a steam engine governor S_1 could be, for example, the variable load of the engine, and S_2 the speed of rotation. By means of a centrifugal governor M_2 the opening of a steam valve is adjusted as a function of the engine speed, thus modifying the relation between load S_1 and speed S_2 in the direction of control. Equations 1 and 2 also apply in this case. Employing already existing terminology we can refer to the type of control arrangements under discussion as continuous indirect controllers. Let us now briefly describe the static behaviour of such controllers.

Suppose that S_1 varies by a small amount ΔS_1 such that S_2 varies by ΔS_2. If the variations are sufficiently small we have, from Eq 1

$$S_2 + \Delta S_2 = \left(A + \frac{\partial A}{\partial S_1}\Delta S_1 + \frac{\partial A}{\partial S_2}\Delta S_2 \right)(S_1 + \Delta S_1)$$

or

$$\frac{\Delta S_2}{S_2}\left(1 - S_1\frac{\partial A}{\partial S_2}\right) = \frac{\Delta S_1}{S_1}\left(1 + \frac{S_1}{A}\frac{\partial A}{\partial S_1}\right) \tag{3}$$

Let us introduce the notation

$$1 + \frac{S_1}{A}\frac{\partial A}{\partial S_1} = K$$

$$-\frac{S_1}{A}\frac{\partial A}{\partial S_2} = k \tag{4}$$

where, according to the above discussion, k is positive in the entire region of control. The value K represents a measure of the non-linearity of the transmission system. In a linear system $K = 1$; in general, K may be greater or smaller than 1.

Introducing the values K and k, Equation 3 becomes

$$\frac{\Delta S_2}{S_2} = \frac{K}{1+k}\frac{\Delta S_1}{S_1} \tag{5}$$

This equation shows the effect of a relative variation $\Delta S_1/S_1$ of variable S_1 on the relative variation $\Delta S_2/S_2$ of S_2. If there were no control action, $k = 0$ and

$$\frac{\Delta S_2}{S_2} = K\frac{\Delta S_1}{S_1} \tag{6}$$

The factor

$$R = \frac{1}{1+k} \tag{7}$$

thus indicates by what fraction the variation of S_2 is reduced by the presence of the control system. The value R can be considered as a measure of the static control accuracy: we shall term it the "control factor".

For telephone [repeater] amplifiers K is very close to 1. Then, for infinitely small variations dS_1 and dS_2 we have, from Eq 5

$$\frac{dS_2}{S_2} = R\frac{dS_1}{S_1}$$

If the control factor is to remain constant over the entire region of control then, by integration we obtain

$$S_2 = CS_1^R \tag{8}$$

where C is an arbitrary constant. This equation represents the controller characteristic for constant control accuracy. As Figure 3 shows, as R decreases, the characteristic is a closer and closer approximation to the ideal of perfect control (dashed line).

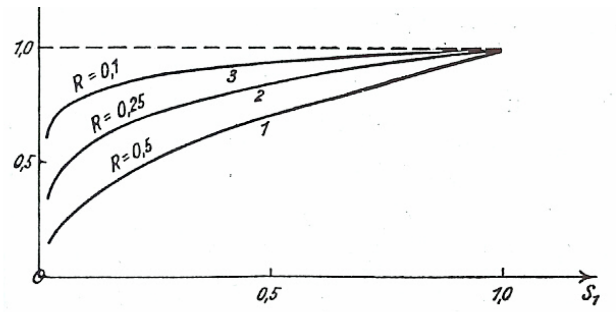

Figure 3. Control characteristics

The discussion so far sheds no light on the stability behaviour, since implicit in Eq 1 and 2 is the assumption that a steady state has been reached. The way in which the time dependency of the variables influences stability can most easily be clarified by the following.

Let the control system be distortionless[3], but introduce a finite time delay t_1. Any variation in S_2 will thus be reflected in a corresponding variation in the transmission factor A, but with a delay equal to t_1. Suppose, for

[3] K Küpfmüller, *ENT*, Vol. 5, 1928, p. 18

example that there is a step change Δ_1 in S_1, then after time t_1 there will be a step change in the value of A of

$$\frac{\partial A}{\partial S_2}\cdot\Delta_1$$

As a result there will be an additional change in the value of S_2 equal to

$$\Delta_2 = S_1\frac{\partial A}{\partial S_2}\Delta_1$$

or, from Eq. 4

$$\Delta_2 = -k\Delta_1 \qquad (9)$$

In order that any initial variation does not increase indefinitely, the condition

$$k \le 1$$

must hold, and from Eq. 7, the control factor must obey the condition

$$R \ge 0.5$$

In other words, the control accuracy in this case can never exceed 50%. Curve 1 in Figure 3 shows the limit of accuracy that can be expected of such a controller. Since in practice far higher control accuracies can be obtained, one suspects that this is connected with the fact that the controller always introduces damping of the process. As will be shown in more detail below, such damping is in fact of decisive influence on the stability of the controller.

3. Stability conditions for the continuous indirect controller

The dynamic behaviour of the continuous indirect controller has been studied in detail in a large number of publications, particularly regarding its application to prime movers. Summaries can be found in Von Mises[4] and W. Hort[5].

The principle of these stability investigations is known as the method of small oscillations. It consists in setting up the differential equations of the system for small deviations from the state of equilibrium. In this way, for a system with n energy-storing components, an nth order differential equation can be derived for each system variable. The eigenoscillations of the system can thus be derived from an equation with constant coefficients a_1, a_2, ... a_n of the form

$$\frac{\mathrm{d}^n S}{\mathrm{d}t^n} + a_1\frac{\mathrm{d}^{n-1}S}{\mathrm{d}t^{n-1}} + a_2\frac{\mathrm{d}^{n-2}S}{\mathrm{d}t^{n-2}} + \cdots + a_n S = 0$$

Introducing

$$S = \mathrm{e}^{pt}$$

where p denotes the complex frequency, we obtain the characteristic equation

$$p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_n = 0 \qquad (10)$$

The condition for the existence of a stable state of equilibrium is then that no root of this equation has a positive real part. After a theorem of Hurwitz this condition can be formulated as follows: all the coefficients a_1 to a_n must be positive or zero; further the following conditions must also hold:

$$\begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix}, \text{etc} > 0$$

This theorem is of great value for the treatment of stability problems when the system in question has only a few degrees of freedom. For controllers in amplifier technology, however, the number of degrees of freedom is often so large that this method leads to extremely complicated results. Neither does this theorem give any information on the course of events. There is, however, another approach that can give an insight into the dynamic behaviour of the type of controllers under consideration, even in very complicated cases.

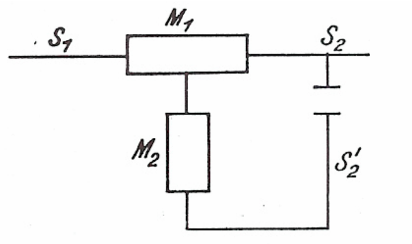

Figure 4. Feedback path [opened]

The control arrangement can be viewed as a feedback system. The feedback path is indicated by the arrow in Figure 2. The condition for the absence of self-excitation according to familiar rules is derived in the following way. The feedback path is cut and the variable S_2 introduced separately in some other way. In the case of the electrical controller a voltage $S_2' = S_2$ would have to be supplied (Figure 4). If a sinusoidal oscillation with angular frequency ω and small amplitude Δ_1 is now superimposed on S_2', then after a sufficient time S_2 will also vary sinusoidally with amplitude Δ_2, where Δ_1 and Δ_2 are related by an expression of the form

$$\Delta_2 = -kU(\omega)\mathrm{e}^{-iu(\omega)}\Delta_1 \qquad (11)$$

Here $-kU$ is the transmission factor [amplitude ratio] and u the transmission angle [phase shift] of the feedback path; for $\omega = 0$, the quiescent state

$$U = 1,\ u = 0 \qquad (12)$$

[4] *Technische Schwingungslehre*, Berlin, 1922, p.266
[5] *Encykl. der math. Wiss*., Vol. IV, Part 1, 2, p.254

so that

$$\Delta_2 = -k\Delta_1$$

in agreement with Equation 9. For the limiting condition, when the system is just on the point of oscillation then $\Delta_2 = \Delta_1$ in Equation 11, so

$$\sin u(\omega) = 0 \qquad (13)$$

$$kU(\omega) = 1 \qquad (14)$$

The frequency of the oscillation follows from Equation 13, the value k from Equation 14.

Equations 13 and 14 form the familiar conditions for oscillations in a feedback loop.

The general representation of the control process is now obtained when, in an analogue fashion, the so-called indicial response [step response] is introduced instead of the frequency characteristics U and u. To do this we need a lemma, which will now be derived for completeness.

4 Representation of functions by means of impulses

In mathematical physics there is a class of functions that can be represented in terms of sources. An example of such a representation will be explained briefly with reference to Figure 5.

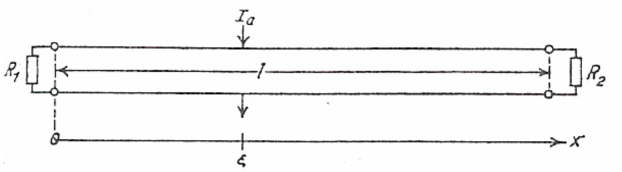

Figure 5. Transmission line with source of interference

Let a transmission line of length l be terminated with arbitrary resistances R_1 and R_2. Assume that at point $x = \zeta$ the line is affected by capacitative interference from some adjacent telephone or high tension cable, giving rise to a current I_a in the line. At a given point x in the line a current I_x will flow, which can be expressed in the form

$$I_x = I_a\, \varphi(x,\zeta) \qquad (15)$$

where the function $\varphi(x, \zeta)$ may be derived easily from transmission line theory. The point $x = \zeta$ can thus be considered to be the origin of a source of strength I_a; the function φ then gives the corresponding current distribution. If the line is subjected over its entire length to an external electric field, then one can imagine different source strengths at each point of the line. In this case we can write:

$$I_a = I(\zeta)\,\mathrm{d}\zeta \qquad (16)$$

where $I(\zeta)$ represents the induced current density at any point $x = \zeta$ on the line. Every such source gives rise to a current distribution according to Equation 15, and the actual current is found by summing the contributions from the individual sources along the length of the line, thus:

$$I_x = \int_0^l \varphi(x,\zeta)\, I(\zeta)\,\mathrm{d}\zeta \qquad (17)$$

In this formula the induced current I_x is thus represented in terms of sources. The function $\varphi(x, \zeta)$ is the "influence function" or the "Green's function" of the boundary value problem.

This method is sometimes used to calculate crosstalk currents or interference from high tension lines in telephone cables[6]. Another example is the calculation of the acoustic fields from spatially extensive sound sources[7].

A completely similar manner of representation can be used for functions of time, rather than space, where the impulse takes the place of the source.

If, in an arbitrary linear transmission system, one system variable is changed suddenly at time $t = \tau$ by a given amount S_1 then a dependent system variable S_2 will also vary according to the expression:

$$S_2 = S_1\varphi(t-\tau) \qquad (18)$$

where the function φ is known as the indicial response of the system.

If the variable S_1 then returns suddenly to its original value at time $t = \tau + \mathrm{d}\tau$ (Figure 6), then it follows from Equation 18 that the behaviour of S_2 is given by

$$S_2 = S_1\varphi(t-\tau) - S_1\varphi(t-\tau-\mathrm{d}\tau)$$

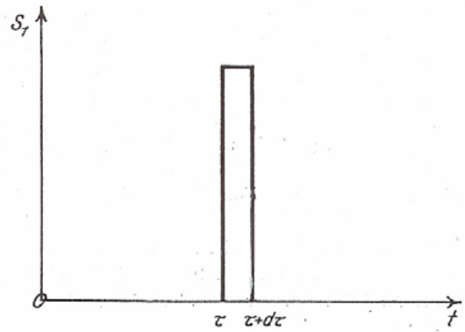

Figure 6. Impulse

If $\varphi(t)$ is a continuous function then if the change in S_1 is of sufficiently short duration, then

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[6] K. Küpfmüller, *Wiss. Veröff. aus dem Siemens-Konzern*, Vol. 1, No. 3, p. 18, 1922; *Archiv für Elektrotechnik*, Vol.12, p. 160, 1923. G. Eggeling, *ENT* Vol. 5, p. 312, 1928

[7] For example: H. Riegger, *Wiss. Veröff. aus dem Siemens-Konzern*, Vol. 3, No. 2, p. 67, 1924. H. Stenzel, *ENT* Vol. 4, p. 239, 1927

$$S_2 = S_1 \, d\tau \, \varphi'(t-\tau) \qquad (19)$$

where $\varphi'(t)$ is the first derivative of the indicial response $\varphi(t)$. The function $\varphi'(t - \tau)$ represents the time response of S_2 to an impulse of the form of Figure 6, if the amplitude of the impulse is $1/\mathrm{d}\tau$. Such an impulse will be termed a unit impulse after G. A. Campbell. It has, strictly speaking, an infinitely short duration and an infinitely great amplitude such that the product of duration and amplitude is equal to unity. If S_1 now varies in an arbitrary manner $S_1(t)$, then the time variation can be divided into narrow strips as shown in Figure 7.

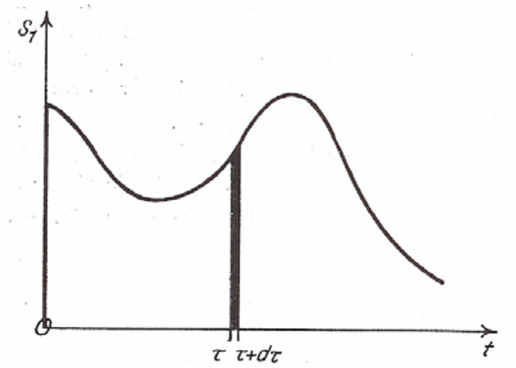

Figure 7. Function divided into impulses

The function $S_1(t)$ can thus be represented by a series of contiguous impulses with an amplitude $S_1(\tau)$ at time τ. According to Equation 19, to each such impulse there corresponds a variation in S_2 given by

$$\mathrm{d}S_2 = S_1(\tau)\varphi'(t-\tau)\,\mathrm{d}\tau \qquad (20)$$

and the value of S_2 itself at arbitrary time t is found by integration over all τ:

$$S_2 = \int_0^t \varphi'(t-\tau)\,S_1(\tau)\,\mathrm{d}\tau \qquad (21)$$

By analogy with Equation 17 we can say that Equation 21 is an impulse representation of S_2. If the indicial response possesses discontinuities of the form

$$\varphi(t)\Big|_{t_1+\varepsilon} = \varphi(t)\Big|_{t_1-\varepsilon} + c_1$$

where c_1 represents a constant, then an impulse of the type of Figure 6 with an amplitude $c_1/\mathrm{d}\tau$ must be added to $\varphi'(t)$ at time $t = t_1$. Equation 21 gives a transformation rule for the calculation of the transient response for any variation in S_1; the technique has been applied in various ways[8].

[8] J. Carson, Proc. Am. In., 1919, p. 407. F. Lüschen and K. Küpfmüller, *Wiss. Veröff. uas dem Siemens-Konzern*, Vol. 3, No. 1, p. 109, 1923. K. Küpfmüller, *ENT*, Vol. 5, p. 18, 1928

In general the indicial response will be zero between time $t = 0$ and a given point t_1 representing the time lag of the system. Then from Equation 21

$$S_2 = \int_0^{t-t_1} \varphi'(t-\tau)S_1(\tau)\,\mathrm{d}\tau \qquad (22)$$

If, in addition, the indicial response reaches its final value at time $t = t_2$, then for $t > t_2$, in place of Equation 22 we have

$$S_2 = \int_{t-t_2}^{t-t_1} \varphi'(t-\tau)\,S_1(\tau)\,\mathrm{d}\tau \qquad (23)$$

We shall denote t_2 as the transition time.

5. The behaviour of the control process

The indicial response of the control system is obtained by investigating the time behaviour of S_2 in Figure 4 when S_2' is suddenly changed by a given amount. In general, for the control arrangements under consideration, the indicial response has the form shown in Figure 8.

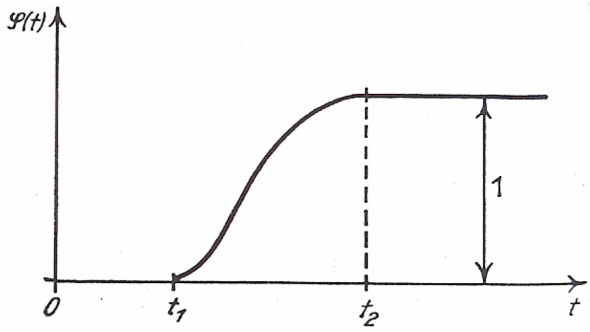

Figure 8. Indicial response

Times t_1 and t_2 represent time lag and transition time; the constant final value is arbitrarily set to 1. The practical advantage of the introduction of the indicial response lies in the fact that the time evolution of this function can easily be determined experimentally or estimated by means of well known approximation rules.

For a small step change Δ_1 in S_1 at time $t = 0$, the change Δ_2 in S_2 follows from Equation 9 as:

$$\Delta_2 = -k\Delta_1\varphi(t) \qquad (24)$$

If S_1 does not undergo a step change, but varies as an arbitrary function y(t), then from Equation 22 we have:

$$\Delta_2 = -k \int_0^{t-t_1} \varphi'(t-\tau)y(\tau)\mathrm{d}\tau \qquad (25)$$

If in Figure 4 in addition to S_2' the variable S_1 also changes by a given small amount, then S_2 will vary by an additional amount $P(t)$. If the feedback loop is now closed again, the quantity Δ_2 in Equation 25 represents the amount by which the variation $y(t)$ of S_2 differs from $P(t)$. Hence

$$y(t) + k \int_{0}^{t-t_1} \varphi'(t-\tau)\, y(\tau)\, \mathrm{d}\tau = P(t) \qquad (26)$$

This is a type 2 linear integral equation which allows the time behaviour of $y(t)$ to be calculated providing that of $P(t)$, the variation of S_2 in the absence of control, is known. Of the various methods for solving this equation we introduce an iterative procedure, which is particularly suited for graphical evaluation.

If $P(t) = 0$ for negative t, then it may be seen easily from Equation 26 that:

$$\left.\begin{aligned}
&t < t_1, \qquad y(t) = P(t) = y_1(t); \\
&t_1 < t < 2t_1, \quad y(t) = \\
&P(t) - k \int_0^{t-t_1} \varphi'(t-\tau) y_1(\tau) \mathrm{d}\tau = y_2(t); \\
&2t_1 < t < 3t_1 \quad y(t) = \\
&P(t) - k \int_0^{t-2t_1} \varphi'(t-\tau) y_1(\tau) \mathrm{d}\tau \\
&\quad - k \int_{t-2t_1}^{t-t_1} \varphi'(t-\tau) y_2(\tau) \mathrm{d}\tau = y_3(t)
\end{aligned}\right\} \qquad (27a)$$

etc

In this manner we obtain the behaviour of $y(t)$ in time steps from t_1 up to time $t = t_2$. After this point in time the following relation holds

$$y(t) = P(t) - k \int_{t-t_2}^{t-t_1} \varphi'(t-\tau)\, y(\tau)\, \mathrm{d}\tau \qquad (27b)$$

from which $y(t)$ can be determined step-by-step by subtraction of the integral expression from $P(t)$.

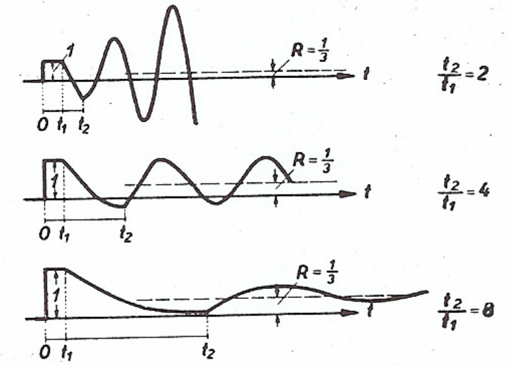

Figure 9. The control process

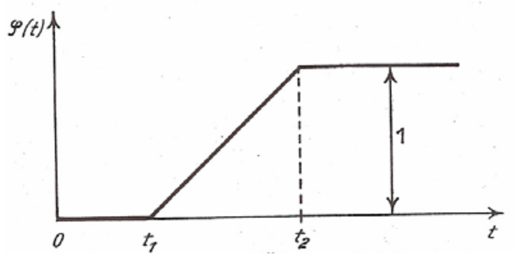

Figure 10. Idealised indicial response

The curves of Figure 9 are derived using this technique, which show the control process for the simple case of the indicial response of Figure 10, where the response rises linearly to its final value. The control factor is assumed to be 1/3. It can be seen that the ratio of the transition time to the time lag, t_2 / t_1, gives a measure of the stability of the controller. For $t_2 / t_1 = 2$ any disturbance leads to a continuously growing oscillation; the case $t_2 / t_1 = 2$ represents neutral equilibrium; while for $t_2 / t_1 = 8$ the disturbance dies gradually away.

This relationship between the speed with which the indicial response reaches its final value, and the stability of the controller, can be expressed in an even more general fashion.

For a free oscillation of the controller $P(t) = 0$ and we have:

$$y(t) + k \int_{t-t_2}^{t-t_1} \varphi'(t-\tau)\, y(\tau)\, \mathrm{d}\tau = 0 \qquad (28)$$

If

$$y(t) = \mathrm{e}^{i\omega t}$$

it follows then that:

$$1 + k \int_{t_1}^{t} \varphi'(v)\, \mathrm{e}^{-i\omega v}\, \mathrm{d}v = 0 \qquad (29)$$

In the limiting case of neutral equilibrium any oscillation, once introduced, will last indefinitely; for very large t, ω is a real value representing the frequency of the oscillation. In this case Equation 29 can only hold if

$$\int_{t_1}^{\infty} \varphi'(v) \sin \omega v \;\, \mathrm{d}\, v = 0 \qquad (30)$$

and

$$-k \int_{t_1}^{\infty} \varphi'(v) \cos \omega v \;\, \mathrm{d}\, v = 1 \qquad (31)$$

Equation 30 gives the frequency of the oscillation and Equation 31 a limiting value for k.

From Equation 7 we obtain the control factor representing the boundary between stable and unstable regions for the controller; it is:

$$R_0 = \frac{\int_{t_1}^{\infty} \varphi'(v) \cos \omega\, v \,\mathrm{d}\, v}{\int_{t_1}^{\infty} \varphi'(v) \cos \omega\, v \,\mathrm{d}\, v - 1} \qquad (32)$$

R_0 represents the control accuracy that can be obtained in the most favourable case for the given indicial response. In practice the control accuracy must be significantly poorer, in order for a disturbance to die away in a sufficiently short time.

Equations 30 and 31 correspond fully to the conditions 13 and 14 of Section 3; they can be derived from the latter using the Fourier integral theorem.

If the indicial response takes the form shown in Figure 10 then from Equation 30

$$\omega = \frac{2\pi}{t_1 + t_2};$$

and from Equation 32 we have

$$\left. \begin{aligned} R_0 &= \frac{(\theta+1)\sin\dfrac{\pi(\theta-1)}{(\theta+1)}}{(\theta+1)\sin\dfrac{\pi(\theta-1)}{(\theta+1)} + \pi(\theta-1)} \\ \theta &= \frac{t_2}{t_1} \end{aligned} \right\} (33)$$

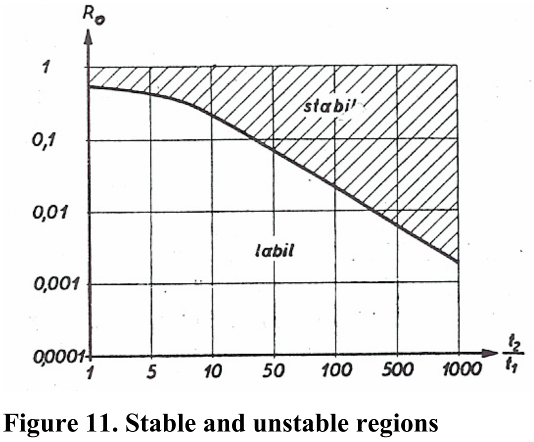

Figure 11. Stable and unstable regions

The critical control factor R_0 is thus a function of t_2/t_1 only. Figure 11 gives a graphical representation of the relationship expressed in Equation 33; it shows, for example, that the transition time must be at least 200 times the time delay if the control accuracy is to achieve 1%. In the absence of a pure time delay ($t_1 = 0$) then any control accuracy can be achieved.

6. Approximation rules

It is noteworthy that the magnitude of the critical control factor is relatively independent of the precise form of the indicial response.

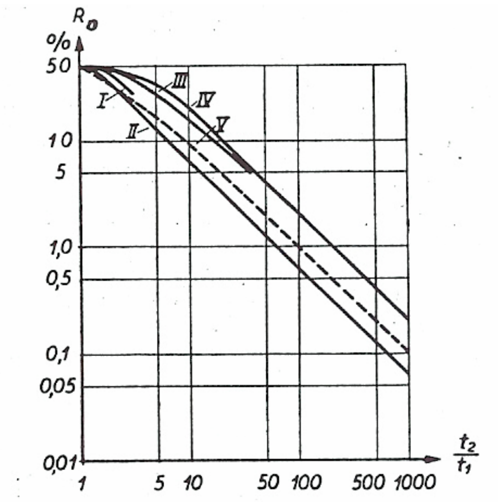

Figure 12. Critical control factor for the indicial responses

Figure 12 shows R_0 for the indicial response curves of Figure 13. In I the indicial response consists of two parabolas; in III of straight lines and parabolas; in II ω [sic] forms an exponential function for values of $t > t_1$; IV is equivalent to Figure 10. The time delay and transition time are in each case defined by the tangents at the points of maximum slope of φ, as can be seen in Figure 13.

The dashed curve in Figure 12 shows the case where

$$R_0 = \frac{t_1}{t_1 + t_2}$$

It can be seen that this curve is an approximation to the critical control factor, so we can state the following rule of thumb for the estimation of stability: *The control factor of a continuous indirect controller must be greater than the ratio of time delay to transition time*.

If a controller has a tendency to oscillation, then the transition time must be increased with respect to the time delay.

It can be seen from Figure 9 that after a relatively short time the response becomes a damped sinusoidal oscillation. One can interpret this as the higher harmonics decay significantly faster than the fundamental frequency. The damping factor δ for the fundamental can be calculated from Equation 28, by determining the ratio of amplitudes of two successive half cycles.

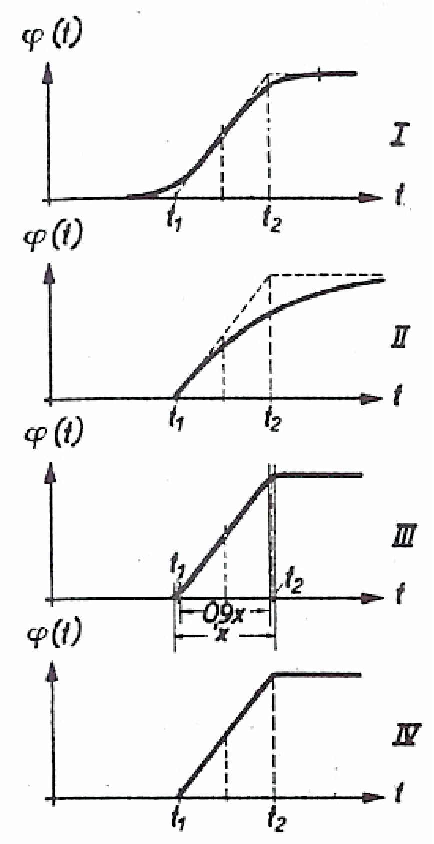

Figure 13. Various indicial responses

In this way the following approximation formula can be obtained:

$$\delta = \frac{1}{T} \ln \frac{R}{R_0} \qquad (34)$$

where

$$T = \frac{t_1 + t_2}{2}$$

R_0 is the critical control factor, and R the actual control factor of the controller. The frequency of the fundamental oscillation is approximately

$$\omega = \frac{\pi}{T}$$

The last relationship holds true for the case where $\varphi'(t)$ is symmetric about the ordinate $t = T$. Thus we have the following rule of thumb for the speed of the control process:

The time required by a controller to move from one state of equilibrium to another is proportional to the arithmetic mean of the transition time and the time delay, and inversely proportional to the logarithm of the ratio of control factor to critical control factor.

As a final application of these rules consider now the case when the control arrangement includes a long filter network. If ω_1 and ω_2 are the limits of the pass band, and n is the order of the filter (in the simplest case thus the number of elements) then from known approximation formulae[9]:

$$T = \frac{2n}{\omega_2 - \omega_1}$$

If n is sufficiently large, then for the rise time

$$t_2 - t_1 = \frac{2\pi}{\omega_2 - \omega_1}$$

and we have

$$\frac{t_1}{t_2} = \frac{2n - \pi}{2n + \pi}$$

For long ladder networks, then ($n >> 1$) the time delay is almost equal to the transition time, and the critical control factor $R_0 \approx 0.5$. Hence when greater control accuracy is required, long ladder networks cannot be used in a control system.

7. Summary

The control arrangements used in amplifier technology are based on the principle of the continuous indirect controller. In order to investigate the stability behaviour of such controllers it is useful to introduce the concept of the indicial response. An integral equation can then be set up which allows the evolution of the control process to be calculated in a simple fashion. Examples of the application of this equation include a determination of the general relationship between stability and speed of control. Stability is greater, the slower the indicial response is in reaching its final value. Approximation rules have been derived for this relationship, and it is demonstrated that the use of long ladder networks in such controllers must in general lead to self-oscillations.

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[9] H. F. Mayer, *ENT,* Vol. 2, p. 335, 1925. K. Küpfmüller, *ENT,* Vol 1, p. 141, 1924