On the General Theory of Governors *(Sur la théorie génerale des régulateurs)*

Monsieur WISCHNEGRADSKI,

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The object of this paper is to investigate the equation of motion of a direct-acting governor mounted on an engine, when the equilibrium between the driving force and the load on the engine is disturbed. The author considers the case in which speed variations and governor displacements are small.

The author expands the various functions expressing the problem as series in increasing powers of small quantities, and considers only terms proportional to the first powers of these quantities. Treating the problem in other respects from a general point of view, he assumes the governor to be provided with a cataract[1] (a piston immersed in a liquid which it displaces as it moves). The resistance of this cataract is assumed to be proportional to piston velocity, which is always relatively small. Finally, the passive friction of the governor and the elements it controls are disregarded, which is permissible for well-designed governors such as those on the various Corliss engines.

Let

t = time elapsed since the disturbance of the equilibrium between driving force and load;

u = the value at time t of the displacement of a point on the governor controlling the element used to vary the driving force; this displacement is assumed to be rectilinear; the velocity of the given point is du/dt and its acceleration d^2u/dt^2;

ω_0 = angular velocity of the engine shaft under normal operating conditions;

ω = shaft velocity at time t;

ω_u = shaft velocity corresponding to governor equilibrium at a distance u from its normal position;

p = normal magnitude of the driving force and load, referred to the same torque arm ρ from the axis;

q = value taken by the load after its sudden variation;

P = magnitude of the driving force corresponding to the position of the governor after time t;

J = moment of inertia of the engine about the shaft.

The author expresses that part of the acceleration d^2u/dt^2 that depends on the speed change of the engine by $K(\omega-\omega_0)/\omega_0$, and that part resulting from the effect of the cataract by $-M\,du/dt$. Furthermore, he sets

$$\frac{K(\omega-\omega_0)}{\omega_0} = Nu; \qquad (p-P)\rho = Lu$$

[1] 'Cataract' was a common term at that time for a hydraulic damper or dashpot. (Translator's note)

where *K, L, M* and *N* are positive constant coefficients whose values depend on the governor design and its arrangement on the engine. The following equation is thus obtained:

$$\frac{d^3u}{dt^3} + M\frac{d^2u}{dt^2} + N\frac{du}{dt} + \frac{KL}{J\omega_0} = \frac{K(p-q)\rho}{J\omega_0}$$

As this equation is third-order, linear and with constant coefficients, its solution depends on the roots of the equation

$$\theta^3 + M\theta^2 + N\theta + \frac{KL}{J\omega_0} = 0$$

To simplify the discussion of the various cases that arise, the author introduces two new variables x and *y.* By setting

$$M = x\sqrt[3]{\frac{KL}{J\omega_0}}, \quad N = y\sqrt[3]{(\frac{KL}{J\omega_0})^2}$$

it is proved that

(1) all governors for which $xy < 1$, when disturbed from their equilibrium position, oscillate with an amplitude which increases indefinitely with time; this renders them unsuitable for governing engine motion;

(2) all governors for which

$$xy > 1, \quad x^2y^2 - 4(x^3 + y^3) + 18xy - 27 < 0$$

oscillate with decreasing amplitude, and their position converges indefinitely to that corresponding to equilibrium;

(3) all governors for which

$$x^2y^2 - 4(x^3 + y^3) + 18xy - 27 > 0$$

do not oscillate, but move in one direction only, converging to the position of equilibrium between the driving force and the new value of the load.

In order to illustrate these principal results more clearly, the author plots the curve given by the equation

$$x^2y^2 - 4(x^3 + y^3) + 18xy - 27 = 0$$

which will be referred to in the following discussion by (A), and the rectangular hyperbola with the equation $xy = 1$. The area between the positive axes *OX* and *OY* is divided by these two curves into three regions.[2] The first, between the hyperbola and the axes, contains those points corresponding to governors with periodic motion of increasing amplitude with time. The second, between the hyperbola and the curve *A,* corresponds to governors with periodic motion but decreasing amplitude. The third, bounded by the curve *A,* corresponds to governors which, once disturbed from their equilibrium position, move in one direction only. If, for a given governor, values of x and *y* are calculated, the nature of the motion of the governor may be determined immediately from the diagram.

[2] The full versions of the paper, published in Russian, German and French over the following two years, include the diagram reproduced as Fig 1 (which shows additional, modern, s-plane plots indicating the characteristic pole configurations associated with various regions).
(Translator’s note)

The author draws the following conclusions.

(1) A governor not provided with a cataract, whatever the other details of its design, cannot operate effectively, since for such a governor M = 0 and consequently x = 0; the condition *xy*> 1 cannot be satisfied.[3]

(2) An isochronous governor[4] cannot operate effectively, even if provided with a cataract of any size whatsoever, since for such a governor *N* = 0, *Y* = 0 and the condition *xy*> 1 again cannot be satisfied.

By considering particular cases, the author demonstrates that the second conclusion is valid for all cataracts, whatever the exponent of the power law relating opposing forces to velocity. Although eliminating *strictly* isochronous governors, the author shows that the search for *very nearly* isochronous governors is still of great importance. Since the limit to which the engine speed tends is

$$\omega_0\left[1+\frac{N(p-q)\rho}{KL}\right]$$

for a correctly operating governor, the final engine speed varies with load less and less as *N* approaches zero, so that it is profitable to make *N* as small as possible[5], increasing cataract action so as to satisfy the condition *xy*> *1.*

Finally, the author explains that, owing to the effect of passive friction, certain governors not satisfying the condition *xy*> t can still act effectively. However, this advantage is obtained at the expense of sensitivity which may only be maintained by reducing passive friction to a minimum.

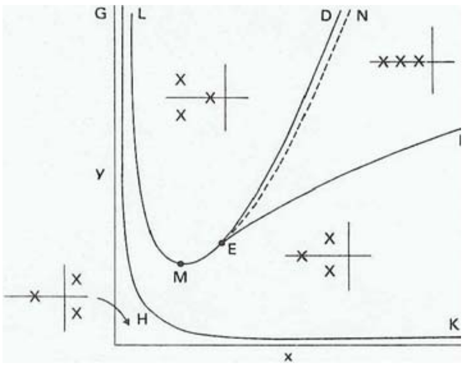

Figure 1. Wischnegradski's stability diagram, with the addition of typical s-plane pole positions.

[3] This conclusion caused some disbelief at the time, since many governors operated effectively without hydraulic damping, owing to the effects of other frictional forces.
(Translator's note)

[4] That is, one without offset. Wischnegradski modelled the engine as an integrator, so an isochronous governor, with integral action, would render the closed-loop system absolutely unstable in the absence of any device to introduce some phase lead. Maxwell (1868) had earlier shown that offset could sometimes be eliminated without instability. See Bennett, S., *A History of Control Engineering 1900-1980*, Stevenage UK, Peter Peregrinus, 1979, and Fuller, A. T. (ed), *Stability of Motion*, London, Taylor & Francis, 1976 for a discussion of nineteenth-century attempts to eliminate offset in steam engine governors.
(Translator's note)

[5] Or, as we might put it now, increasing controller gain reduces steady-state error, so long as the loop remains stable. N is the restoring force constant of the governor, so 1/N corresponds to controller gain.
(Translator's note)